SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2013
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #3

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each Section (A, B, and C) is to be returned in a separate bundle. Multiple choice questions are to be answered on the answer sheet provided.
- All necessary working should be shown in every question, except multiple choice.

Total Marks – 72

- Attempt questions 1 – 13.
- All questions are not of equal value.
- Unless otherwise directed give your answers in simplest exact form.

Examiner: *A.M.Gainford*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \; x > 0$

Multiple Choice

ANSWER ON THE ANSWER SHEET PROVIDED

In Questions 1 to 7 indicate which of the answers A, B, C, or D is the correct answer. Write the letter corresponding to the answer on the answer sheet supplied.

Question 1 (1 mark)

Marks

The derivative of $\sin^2 x =$ **1**

A: $2\sin x$

B: $2\cos x$

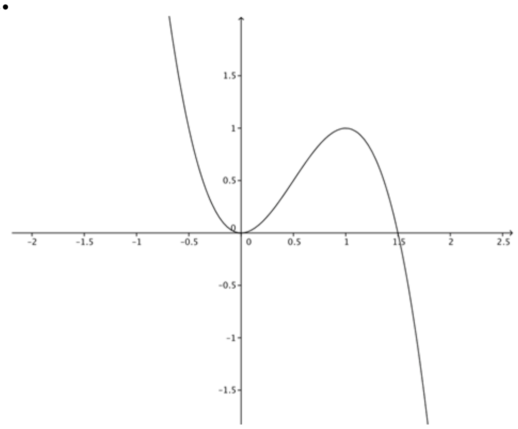
C: $2\sin x\cos x$

D: $2\cos^2 x$

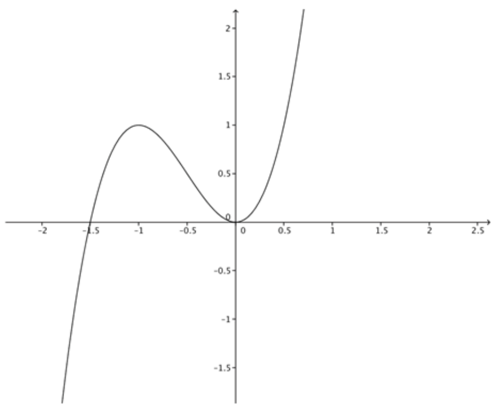
Question 2 (1 mark)

Which of the following best represents the graph of $f(x) = 2x^3 - 3x^2$? **1**

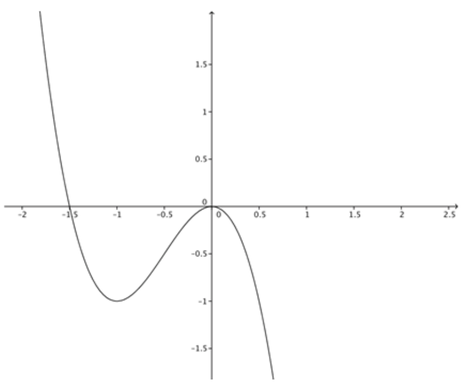
A:



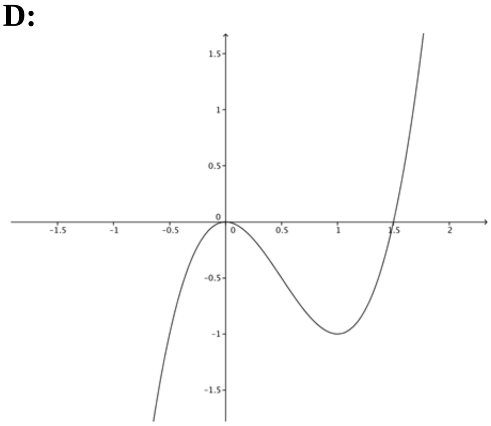
B:



C:



D:



Question 3 (1 mark)

For which values of *x* is the curve $y = x^3 + 2x^2$ concave up? **1**

A: $x < -\frac{2}{3}$

B: $x < -\frac{3}{2}$

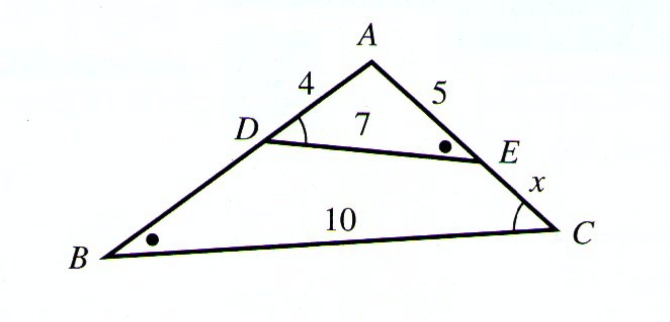
C: $x > -\frac{2}{3}$

D: $x > \frac{3}{2}$

Question 4 (1 mark)

1

In the diagram below, $\angle AED = \angle ABC$ and $\angle ADE = \angle ACB$. The value of *x* is:



A: $\frac{5}{7}$

B: $3\frac{1}{2}$

C: 5

D: 10

Question 5 (1 mark)

What is the greatest value of the function $y = 6 - 3\cos 2x$? **1**

A: 6

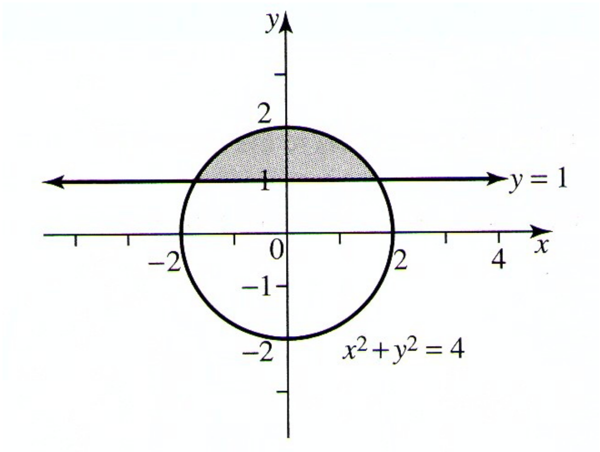
B: 3

C: 0

D: 9

Question 6 (1 mark)

The shaded region between the circle $x^2 + y^2 = 4$ and the line $y = 1$ is shown below. **1**
The area of the region is given by:



A: $\int_{-1}^{1}\left(\sqrt{4-x^2}-1\right)dx$

B: $\int_{-1}^{1}\left(\sqrt{4-x^2}\right)dx$

C: $\int_{-\sqrt{3}}^{\sqrt{3}}\left(\sqrt{4-x^2}\right)dx$

D: $\int_{-\sqrt{3}}^{\sqrt{3}}\left(\sqrt{4-x^2}-1\right)dx$

Question 7 (1 mark)

The second derivative of $\dfrac{e^x + e^{-x}}{2}$ is:- **1**

A: $\dfrac{e^x - e^{-x}}{2}$

B: $\dfrac{e^{2x} + e^{-2x}}{2}$

C: $\dfrac{e^x + e^{-x}}{2}$

D: $\dfrac{e^{2x} - e^{-2x}}{2}$

Section A (20 Marks)
(Start a new booklet)

Question 8 (12 marks)

Marks

(a) Differentiate the following: **5**

(i) $y = 3\sin 2x$

(ii) $y = \dfrac{1}{e^{2x}}$

(iii) $y = x^2 \cos x$

(iv) $y = \dfrac{\ln x}{x}$

(v) $y = \tan^2 3x$

(b) Find **4**

(i) $\int \sin 2x \, dx$

(ii) $\int_0^{\frac{\pi}{4}} \sec^2 x dx$

(iii) $\int \dfrac{2x}{x^2 - 7} dx$

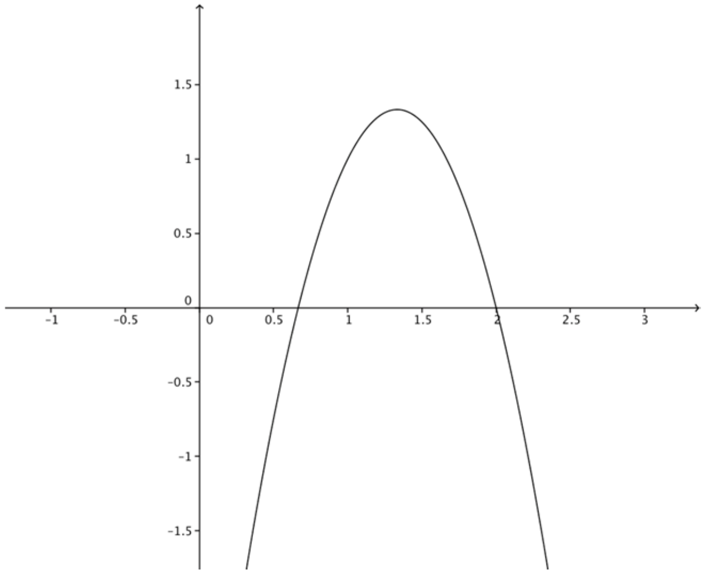
(c) Use Simpson’s rule with five function values to find an approximation to **3**

$$\int_0^4 \frac{x^2}{x+1} dx$$

(Answer in simplified improper fraction form.)

Question 9 (8 marks)

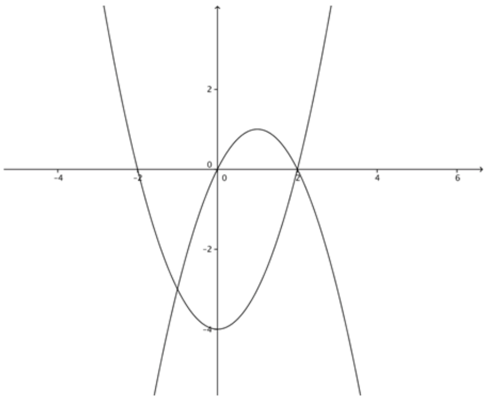
(a) The graph shows $f'(x)$, the derivative of $y = f(x)$. **4**



(i) Copy the graph to your answer booklet.

(ii) Sketch on the graph a possible graph of $y = f(x)$ given that the curve passes through the origin. On your graph of $y = f(x)$ mark any points of inflexion.

(b)



The sketch shows the curves $y = x^2 - 4$ and $y = 1 - (x-1)^2$. **4**

(i) Find the points of intersection of the curves.

(ii) Calculate the area of the region between the two curves.

Section B (23 Marks)
START A NEW BOOKLET

Question 10 (11 Marks)

Marks

(a) Sketch the graph of $y = e^x - 2$ showing the important features. **2**

(b) Find $\int_0^{\frac{\pi}{9}} \sec^2 3x \, dx$ **2**

(c) Sketch the curve $y = 1 + \cos 2x$ in the domain $0 \le x \le 2\pi$. **2**

(e) (i) Sketch the graph of $y = 1 + \ln x$, and shade the area bounded by the curve, the *y*-axis, and the lines $y = 1$ and $y = 2$. **5**

(ii) Make *x* the subject of the equation $y = 1 + \ln x$.

(iii) Find the volume (in terms of *e*) generated by rotating the shaded region about the *y*-axis.

Question 11 (12 Marks)

(a) Consider the function $y = x\log_e x$. **4**

(i) Find the derivative.

(ii) Hence find the minimum value of $x\log_e x$ and justify your answer.

(b) Given the function $y = (3-x)(x-2)^2$: **6**

(i) Find the co-ordinates of the stationary points, and determine their nature.

(ii) Find the co-ordinates of any points of inflexion.

(iii) Sketch the curve in the domain $0 \le x \le 4$.

(c) Find $\frac{d}{dx}\left(\ln(\cos x)\right)$. Give your answer in simplified form. **2**

Section C (22 Marks)

START A NEW BOOKLET

Question 12 (10 Marks)

(a) For the triangle with vertices $P\left(-1,\tfrac{1}{2}\right)$, $Q(1,4)$, and $R(3,1)$: **5**

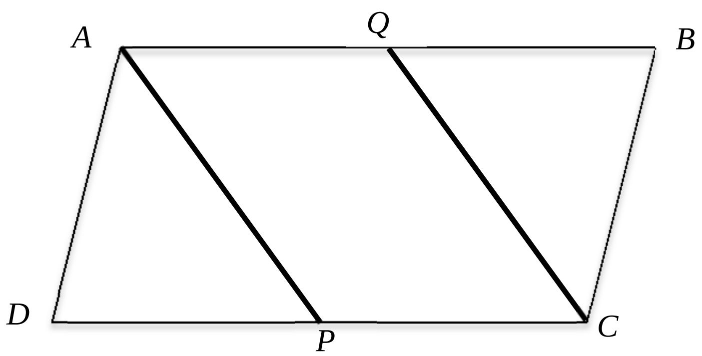
(i) Sketch the triangle on a number plane in your answer booklet.

(ii) Find the midpoint, M, of the interval joining QR.

(iii) Find the gradient of PM.

(iv) Show that PM is the perpendicular bisector of QR.

(b) **5**



The figure $ABCD$ is a parallelogram. AP bisects $\angle DAB$, and CQ bisects $\angle BCD$.

(i) Prove that $\Delta DAP \equiv \Delta BCQ$.

(ii) Prove that $AQ = CP$.

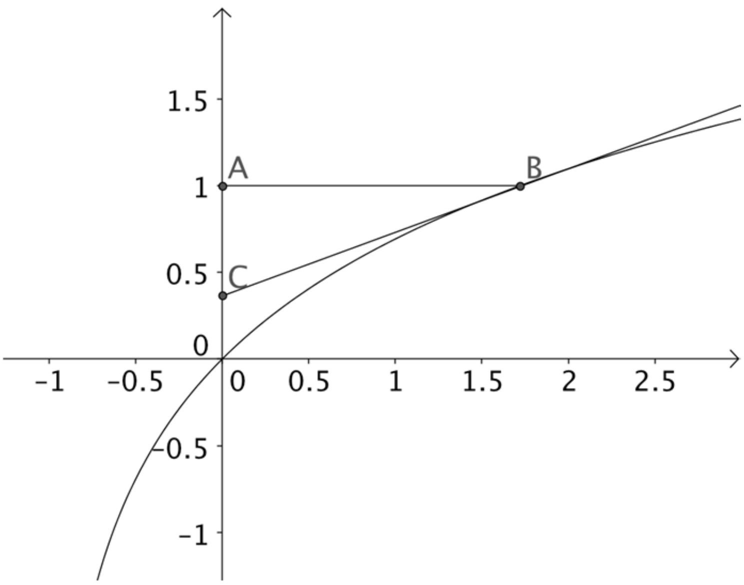
Question 13 (12 Marks)

(a) Find the equation of the normal to the curve $y = 2\ln x$ at the point $(e, 2)$. **2**

(b) (i) On the same diagram, sketch the curves $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$ in the first quadrant. Shade the area bounded by the two curves and the ordinate $x = 2$. **5**

(ii) Find the volume generated when this area is rotated about the x-axis.

(c) **5**



In the diagram, the point B whose y-coordinate is 1, lies on the curve $y = \ln(x+1)$. The tangent to the curve at B cuts the y-axis at C. A straight line through B perpendicular to the y-axis meets the y-axis at A.

(i) Show that the x-coordinate of B is $(e - 1)$.

(ii) Show that the equation of the tangent BC is $x - ey + 1 = 0$.

(iii) Find the length of AC in terms of e.

This is the end of the paper.

Maths 2unit Task 3

Multiple Choice

1 C

2 D

3 C

4 A

5 D

6 D

7 C

Q8 (a) (i) $y = 3\sin 2x$

$y' = 3 . \cos 2x . 2$

$= 6 \cos 2x$ 1

(ii) $y = \frac{1}{e^{2x}} = e^{-2x}$

$y' = e^{-2x} . -2$

$= -2e^{-2x} \quad \left(= \frac{-2}{e^{2x}}\right)$ 1

(iii) $y = x^2 \cos x$

$y' = \cos x . 2x + x^2(-\sin x)$

$= 2x\cos x - x^2 \sin x$ 1

(iv) $y = \frac{\ln x}{x}$

$y' = \frac{x . \frac{1}{x} - \ln x . 1}{x^2}$

$= \frac{1 - \ln x}{x^2}$ 1

(v) $y = \tan^2 3x$

$y' = 2\tan 3x . \sec^2 3x . 3$

$= 6\tan 3x . \sec^2 3x$ 1

(b) (i) $\int \sin 2x \, dx$

$= -\frac{1}{2}\cos 2x + c$ 1

(ii) $\int_0^{\pi/4} \sec^2 3x \, dx$

PROBLEM: $y = \sec^2 3x$ has a discontinuity at $x = \frac{\pi}{6}$. Hence it is not possible to calculate $\int_0^{\pi/4} \sec^2 3x \, dx$ 2

If this is not noticed: OR

$\int_0^{\pi/4} \sec^2 3x \, dx$

$= \frac{1}{3} [\tan 3x]_0^{\pi/4}$ 1

$= \frac{1}{3} \{ [\tan \frac{3\pi}{4}] - [\tan 0] \}$

$= \frac{1}{3} \{ -1 - 0 \}$

$= -\frac{1}{3}$

(iii) $\int \frac{2x}{x^2 - 7} \, dx$ 1

$= \ln(x^2 - 7) + c.$

(c) $\int_0^4 \frac{x^2}{x+1} \, dx$ 1

x	0	1	2	3	4
y	0	$\frac{1}{2}$	$\frac{4}{3}$	$\frac{9}{4}$	$\frac{16}{5}$

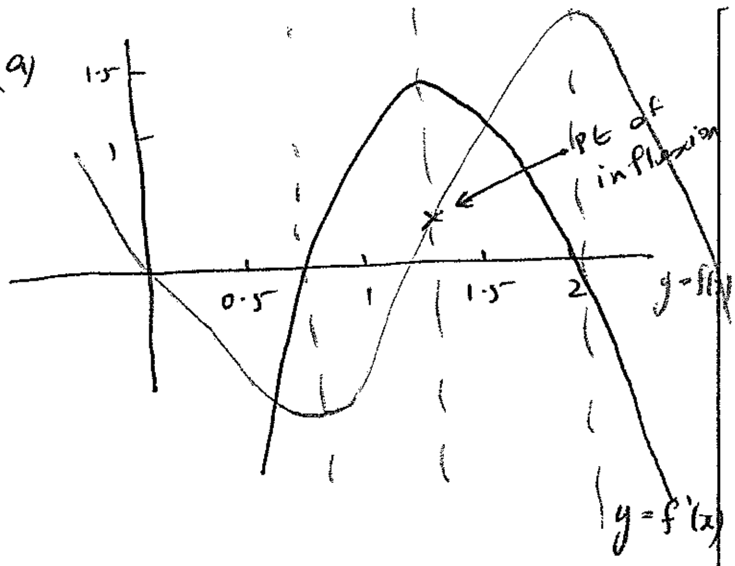
$\approx \frac{1}{3}\left(0 + 4 \times \frac{1}{2} + 2 \times \frac{4}{3} + 4 \times \frac{9}{4} + \frac{16}{5}\right)$ 1

$= \frac{1}{3}\left(2 + \frac{8}{3} + 9 + \frac{16}{5}\right)$

$= \frac{253}{45}$ 1

Q 9

(a)



2 st pts

1 pt of inflexion

1 shape + origin

(b) (i) $y = x^2 - 4 \quad y = 1 - (x-1)^2$

For pt of $\cap$: $x^2 - 4 = 1 - (x-1)^2$

$$x^2 - 4 = 1 - x^2 + 2x - 1$$

$$x^2 - 4 = -x^2 + 2x$$

$$2x^2 - 2x - 4 = 0$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0 \qquad 2$$

$$\therefore x = 2 \text{ or } x = -1$$

$\therefore$ Pts of $\cap$ are $(-1, -3)$ and $(2, 0)$

(ii) Area $= \int_{-1}^{2} \left(1 - (x-1)^2 - (x^2 - 4)\right) dx$

$$= \int_{-1}^{2} (1 - x^2 + 2x - 1 - x^2 + 4)\, dx$$

$$= \int_{-1}^{2} (4 + 2x - 2x^2)\, dx \qquad 2$$

$$= \left[4x + x^2 - \frac{2}{3}x^3\right]_{-1}^{2}$$

$$= \left[8 + 4 - \frac{16}{3}\right] - \left[-4 + 1 + \frac{2}{3}\right]$$

$$= 15 - 6$$

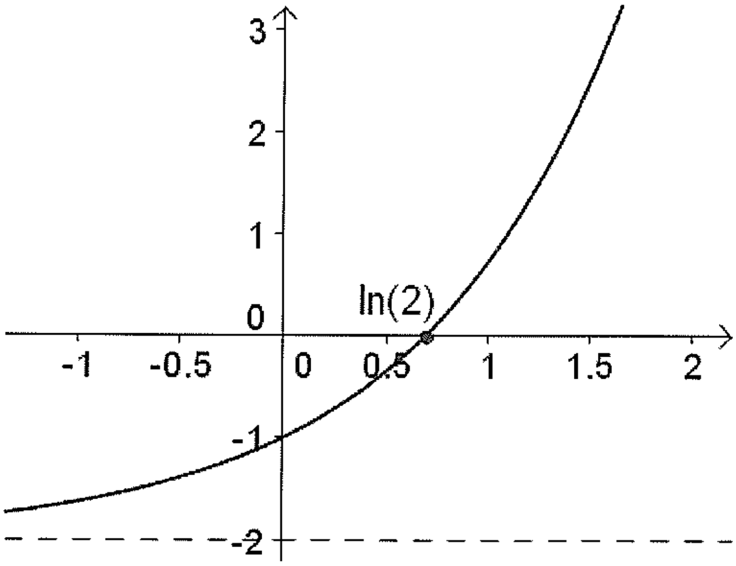
$$= 9 \text{ units}^2$$

2013 Maths 2 unit Task 3

Section B

Question 10

(a)



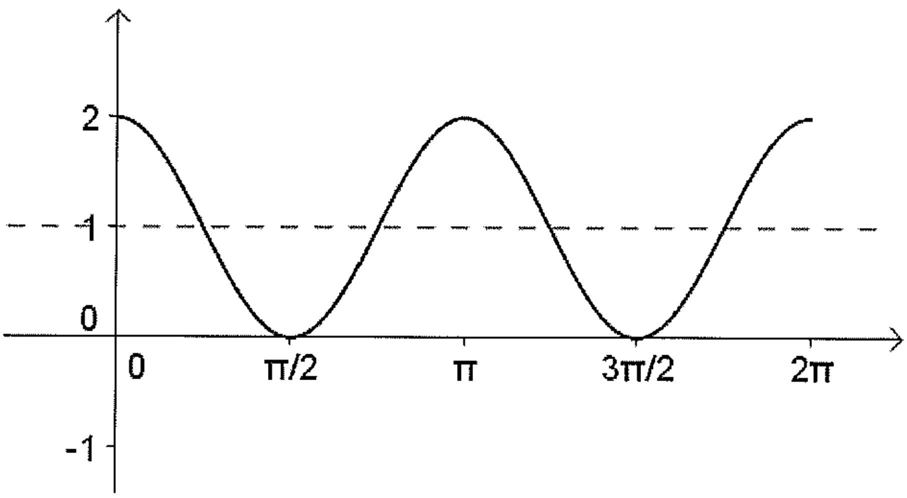
(b)

$$\int_0^{\frac{\pi}{9}} \sec^2 3x \, dx = \frac{1}{3}[\tan 3x]_0^{\frac{\pi}{9}}$$

$$= \frac{1}{3}\left(\tan\left(\frac{\pi}{3}\right) - \tan 0\right)$$

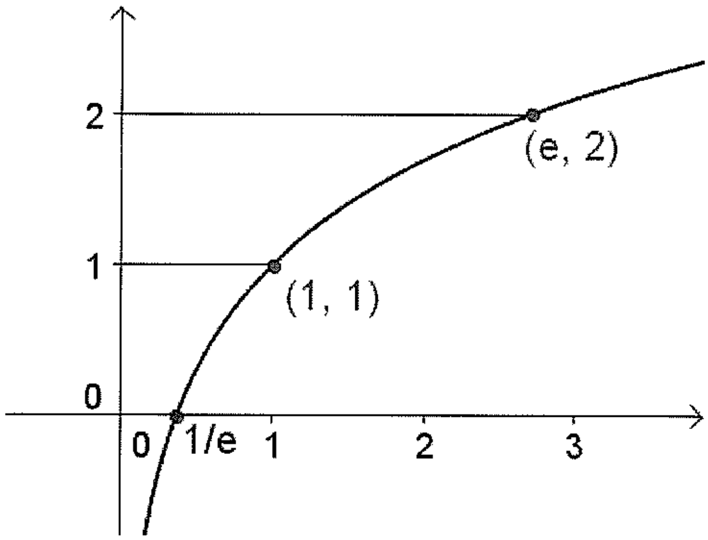
$$= \frac{\sqrt{3}}{3}$$

(c)



(e)

(i)



(ii)

$$y = 1 + \ln x$$

$$y - 1 = \ln x$$

$$e^{y-1} = x$$

(iii)

$$\pi \int_1^2 (e^{y-1})^2 dy = \pi \int_1^2 e^{2y-2} dy$$

$$= \frac{\pi}{2}[e^{2y-2}]_1^2$$

$$= \frac{\pi}{2}(e^2 - 1)\, units^2$$

Q11

(a) (i) $y = x\log_e x$

$y' = \log_e x + 1$

(ii) $y'' = \frac{1}{x}$

Stat pts ($y' = 0$)

$\log_e x + 1 = 0$

$\log_e x = -1$

$x = \frac{1}{e}$.

$y'' = e$ at $x = \frac{1}{e}$

$y'' > 0 \quad \forall x \quad \therefore$ minima

minimum occurs at $x = \frac{1}{e}$

$y = \frac{1}{e}\log_e \frac{1}{e}$

$= -\frac{1}{e}$

(b) (i) $y = (3-x)(x-2)^2$

$y' = -(x-2)^2 + 2(3-x)(x-2)$

$= -x^2 + 4x - 4 + 2(3x - 6 - x^2 + 2x)$

$= -3x^2 + 14x - 16$

Stat Pts $(y'=0)$

$$\frac{(3x-6)(3x-8)}{3}=0$$

$$(x-2)(3x-8)=0$$

$$x=2,\ \frac{8}{3}$$

$$y''=-6x+14$$

Nature at $x=2$

$y''=-12+14=2>0$ minimum.

Nature at $x=\frac{8}{3}$

$y''=-16+14=-2<0$ maximum

At $x=2$, $y=0$

At $x=\frac{8}{3}$, $y=\frac{4}{27}$

$(2,0)$ minimum $\left(\frac{8}{3},\frac{4}{27}\right)$ maximum.

(ii) $y''=-6x+14$

$$-6x+14=0$$

$$x=\frac{7}{3}$$

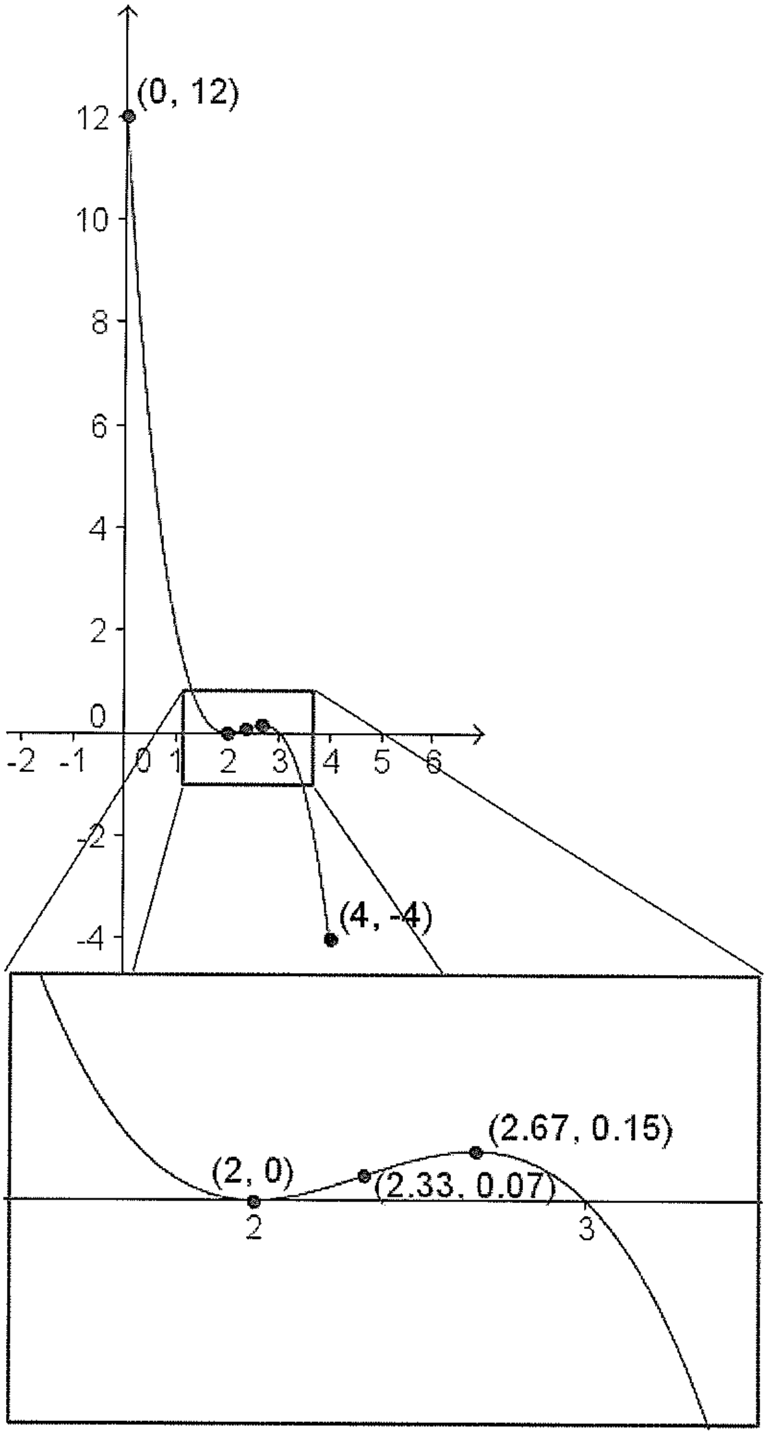
At $x=\frac{7}{3}$, $y=\frac{2}{27}$

Inflexion $\left(\frac{7}{3},\frac{2}{27}\right)$

(iii)

At $x = 0, y = 12$

At $x = 4, x = -4$



(c)

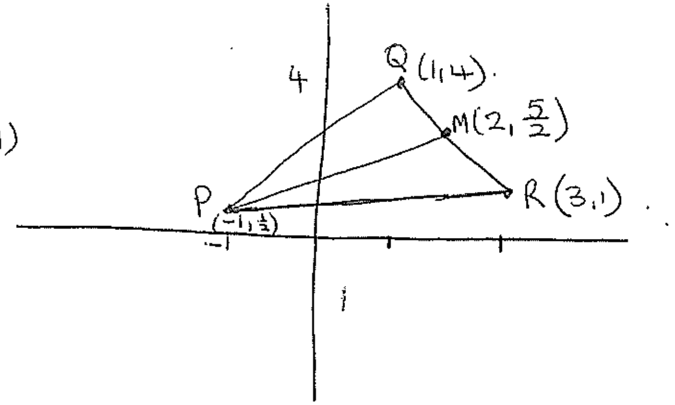
$$\frac{d}{dx}\ln(\cos x) = -\frac{\sin x}{\cos x} = -\tan x$$

2013 2U TASK 3U Q12+13.

(12)

a)

(i)



(1)

(ii)

$$MP_{QR} = \left(\frac{1+3}{2}, \frac{4+1}{2}\right) = \left(2, \frac{5}{2}\right)$$ (1)

(iii)

$$m_{PM} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{5}{2} - \frac{1}{2}}{2+1} = \frac{2}{3}.$$ (1)

(iv)

$M\left(2, \frac{5}{2}\right)$ bisects QR. ⚠ (2)

$$\text{Gradient} = m_{QR} = \frac{4-1}{1-3} = -\frac{3}{2}.$$

⚠

$m_{PM} = -\frac{1}{m_{QR}}$ ∴ perpendicular bisector.

b) In △DAP & BCQ

A: ∠DAP = ∠BCQ (opp ∠'s of //gram are equal then bisected).

S: AD = BC (opp sides of //gram are equal).

A. ∠APD = ∠QBC (opp ∠s of //gram are equal).

△DAP ≡ △BCQ.
AAS.

(3)

By corresp. sides of congruent △'s.

QB = DP.
AB = DC (opp sides of //gram equal)

∴ AB − QB = DC − DP.

(2)

AC = CP QED.

(13) a) $y = 2\ln x$ $\qquad \frac{dy}{dx} = \frac{2}{x}$ at $(e, 2)$.

Perpendicular $= -\frac{dx}{dy} = \frac{-x}{2}$.

(2)

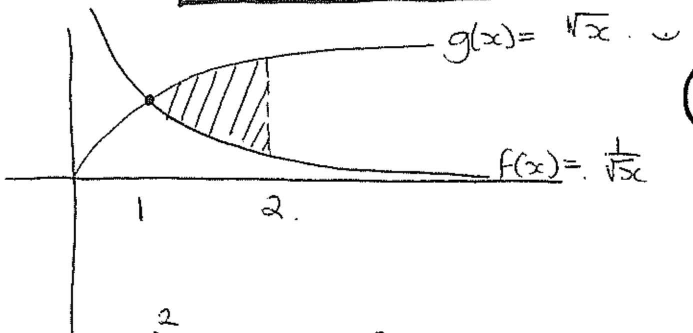
$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{e}{2}(x - e)$$

$$\boxed{y = \frac{e^2}{2} - \frac{ex}{2} + 2}$$

$$\boxed{ex + 2y - e^2 - 4 = 0}$$

b)



(2)

$$V = \pi\int_1^2 g(x)^2 - f(x)^2 \, dx$$

$$= \pi\int_1^2 x - \frac{1}{x} \, dx$$

$$= \pi\left[\frac{x^2}{2} - \ln x\right]_1^2$$

(3)

$$= \pi\left[\left(\frac{4}{2} - \ln 2\right) - \left(\frac{1}{2} - \ln 1\right)\right]$$

$$= \pi\left[1\tfrac{1}{2} - \ln 2 + \ln 1\right] = \pi\left[\frac{3}{2} - \ln 2\right] \text{units}^3.$$

13 (c)

$$y = \ln(x+1)$$

$$1 = \ln(x+1)$$

$$e^1 = e^{\ln(x+1)}$$

$$e = x + 1$$

$$\boxed{x = e - 1}$$

(2)

$$\frac{dy}{dx} = \frac{1}{x+1} = \frac{1}{e} = m$$

$(e-1, 1)$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{e}(x - (e-1))$$

$$ey - e = x + 1 - e$$

(2)

BC is $\boxed{x + 1 - ey = 0}$ ✓

BC cuts y axis when $x = 0$

$$ey = 1$$

$$y = \frac{1}{e}$$

(1)

Co-ords $A(0, 1)$ $\quad C\left(0, \frac{1}{e}\right)$

Distance $\boxed{AC = 1 - \frac{1}{e} \text{ units}}$